1 (12 points) Determine whether the series is convergent. Justify your answers.

(a) (6 points) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2+7}$

Use the Alternating Series Test:

1. It alternates, $\dfrac{n}{n^2+7} > 0$

2. $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+7} = 0$

3. Let $f(x) = \dfrac{x}{x^2+7}$. *Then* $f'(x) = -\dfrac{x^2-7}{(x^2+7)^2} < 0$ *for* $x > \sqrt{7}$.

Thus the sequence is decreasing for $n \geq 3$.

The series converges by the AST.

(b) (6 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-7)^{n+1}}{2^{3n}}$

$$\sum_{n=0}^{\infty} \frac{(-7)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} -7 \cdot \frac{(-7)^n}{(2^3)^n} = \sum_{n=0}^{\infty} -7 \cdot \frac{(-7)^n}{8^n}$$

This is geometric with $a = -7$ *and* $r = -7/8$.

Since $-1 < r < 1$, *it converges to* $\dfrac{-7}{1+7/8} = -\dfrac{56}{15}$.

2 (8 points) Determine whether the statement is true or false. There is no partial credit on this problem.

(a) (2 points) If $\{a_n\}$ is a sequence with limit 5 then $\sum_{n=1}^{\infty}(a_n - 5)$ converges.

False, consider $\sum_{n=1}^{\infty}\frac{5n+1}{n}$.

(b) (2 points) If $\sum_{n=0}^{\infty} a_n$ is divergent then $\sum_{n=0}^{\infty} |a_n|$ is also divergent.

True, if $\sum_{n=0}^{\infty} |a_n|$ *were convergent, then* $\sum_{n=0}^{\infty} a_n$ *would be absolutely convergent.*

(c) (2 points) If $\sum_{n=1}^{\infty} c_n\, 3^n$ diverges then $\sum_{n=1}^{\infty} c_n\,(-4)^n$ also diverges.

True, this is saying that $\sum_{n=1}^{\infty} c_n\, x^n$ *has a radius of convergence that is* ≤ 3.

(d) (2 points) $\sum_{n=1}^{\infty} n^{-\ln(3)}$ converges.

True, this is a p-series with $p = \ln(3) > 1$.

3 (14 points) Consider the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+3)^2\,7^n}$.

(a) (7 points) Find the interval of convergence. Do not check the endpoints.

Use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+4)^2\,7^{n+1}}\cdot\frac{(n+3)^2\,7^n}{(x-3)^n}\right| &= \lim_{n\to\infty}\frac{1}{7}\cdot\left(\frac{n+3}{n+4}\right)^2|x-3| \\ &= \frac{1}{7}\cdot|x-3|\end{aligned}$$

Setting $\frac{1}{7}\cdot|x-3|<1$ *gives the interval* $-4<x<10$.

(In fact, this one converges at both endpoints.)

(b) (7 points) How many terms should we use to estimate $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+3)^2\,7^n}$ to within 0.01?

The series starts with $\frac{1}{9}-\frac{1}{16\cdot 7}+\frac{1}{25\cdot 49}-\cdots$

Since $\frac{1}{16\cdot 7}<\frac{1}{100}$ *we need only one term to get the desired degree of accuracy.*

4 (8 points) Compute the 3^{rd}-degree Taylor polynomial of $f(x) = \sqrt[5]{x}$ centered at $a = -1$.

$$\begin{array}{ll} f(x) = x^{1/5} & f(-1) = -1 \\ f'(x) = \frac{1}{5}x^{-4/5} & f'(-1) = \frac{1}{5} \\ f''(x) = -\frac{4}{25}x^{-9/5} & f''(-1) = \frac{4}{25} \\ f'''(x) = \frac{36}{125}x^{-14/5} & f'''(-1) = \frac{36}{125} \end{array}$$

$$\begin{aligned} f(x) &\approx -1 + \frac{1}{5}(x+1) + \frac{4/25}{2!}(x+1)^2 + \frac{36/125}{3!}(x+1)^3 \\ &= -1 + \frac{1}{5}(x+1) + \frac{2}{25}(x+1)^2 + \frac{6}{125}(x+1)^3 \end{aligned}$$

5 (8 points) Evaluate the indefinite integral $\int x^2 \tan^{-1}(2x)\,dx$ as a power series.

Use $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ *to get* $x^2 \tan^{-1}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{2n+1} x^{2n+3}$.

Then

$$\begin{aligned} \int x^2 \tan^{-1}(2x)\,dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{2n+1} x^{2n+3}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{2n+1} \int x^{2n+3}\,dx \ \textit{(by the theorem on term-by-term integration)} \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2n+1)(2n+4)} x^{2n+4} \end{aligned}$$